On the greatest solution of equations in CLL_R ☆

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Abstract

It is shown that, for any equation $X =_{RS} t_X$ in the LLTS-oriented process calculus CLL_R, if X is strongly guarded in t_X, then the recursive term $\langle X|X = t_X\rangle$ is the greatest solution of this equation w.r.t Lüttgen and Vogler's ready simulation.

Keywords: logic labelled transition system, process calculus, specification, solution of equations

1. Introduction

The notion of logic labelled transition system (LLTS for short), proposed by Lüttgen and Vogler, provides a framework to combine operational and logical styles of specification [2,3,4]. Recently, inspired by this work, we propose an LLTS-oriented process calculus CLL_R, and establish the uniqueness of solutions of equations in CLL_R under a certain circumstance [5]. This note considers solutions of equations in CLL_R furtherly. Firstly, through giving an example, it will be shown that, without the assumption that X does not occur in the scope of any conjunction in t, an equation $X =_{RS} t$ may have more than one consistent solution. Secondly, under the hypothesis that X is strongly guarded in a given term t, it will be shown that the process $\langle X|X = t\rangle$ is the greatest solution of the equation $X =_{RS} t$. This result reveals that $\langle X|X = t\rangle$ captures the loosest specification satisfying the equation $X =_{RS} t$ whenever X is strongly guarded in t. The rest of this note is organized as follows. The next section recalls some related notions and results. The main result will be given in Section 3.

2. Preliminaries

This section will recall a number of related notions and results. Given space limitation, we only list these ones. For details see [2,3,4,5]. We begin with

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recalling the notion of LLTS. Let Act be the set of visible action names ranged over by a, b, etc., and let Act_τ denote $Act \cup \{\tau\}$ ranged over by α and β, where τ represents invisible actions. A labelled transition system with predicate is a quadruple $(P, Act_\tau, \rightarrow, F)$, where P is a set of states, $\rightarrow \subseteq P \times Act_\tau \times P$ is the transition relation and $F \subseteq P$. As usual, we write $p \stackrel{\alpha}{\rightarrow}$ (or, $p \stackrel{\alpha}{\not\rightarrow}$) if $\exists q \in P.p \stackrel{\alpha}{\rightarrow} q$ ($\nexists q \in P.p \stackrel{\alpha}{\rightarrow} q$, resp.). The ready set $\{\alpha \in Act_\tau : p \stackrel{\alpha}{\rightarrow}\}$ of a given state p is denoted by $\mathcal{I}(p)$. A state p is stable if $p \stackrel{\tau}{\not\rightarrow}$. Some useful decorated transition relations are listed below:

(1) $p \stackrel{\alpha}{\rightarrow}_F q$ iff $p \stackrel{\alpha}{\rightarrow} q$ and $p, q \notin F$; (2) $p \stackrel{\epsilon}{\Rightarrow} q$ iff $p(\stackrel{\tau}{\rightarrow})^* q$, where $(\stackrel{\tau}{\rightarrow})^*$ is the transitive and reflexive closure of $\stackrel{\tau}{\rightarrow}$; (3) $p \stackrel{\alpha}{\Rightarrow} q$ iff $\exists r, s \in P.p \stackrel{\epsilon}{\Rightarrow} r \stackrel{\alpha}{\rightarrow} s \stackrel{\epsilon}{\Rightarrow} q$; (4) $p \stackrel{\gamma}{\Rightarrow} |q$ iff $p \stackrel{\gamma}{\Rightarrow} q \stackrel{\tau}{\not\rightarrow}$ with $\gamma \in Act_\tau \cup \{\epsilon\}$; (5) $p \stackrel{\epsilon}{\Rightarrow}_F q$ iff there exists a sequence of τ-transitions from p to q such that all states along this sequence, including p and q, are not in F; the decorated transition $p \stackrel{\alpha}{\Rightarrow}_F q$ may be defined similarly; (6) $p \stackrel{\gamma}{\Rightarrow}_F |q$ iff $p \stackrel{\gamma}{\Rightarrow}_F q \stackrel{\tau}{\not\rightarrow}$ with $\gamma \in Act_\tau \cup \{\epsilon\}$.

Definition 2.1 ([3])**.** An LTS $(P, Act_\tau, \rightarrow, F)$ is an LLTS, if, for each $p \in P$, **(LTS1)** $p \in F$ if $\exists \alpha \in \mathcal{I}(p) \forall q \in P(p \stackrel{\alpha}{\rightarrow} q$ implies $q \in F)$; **(LTS2)** $p \in F$ if $\nexists q \in P.p \stackrel{\epsilon}{\Rightarrow}_F |q$. An LLTS $(P, Act_\tau, \rightarrow, F)$ is τ-pure if, for each $p \in P$, $p \stackrel{\tau}{\rightarrow}$ implies $\nexists a \in Act.\ p \stackrel{a}{\rightarrow}$.

Compared with usual LTSs, one distinctive feature of LLTS is that it involves consideration of inconsistencies. The motivation behind such consideration lies in dealing with inconsistencies caused by conjunctive composition. The predicate F in LLTS is used to denote the set of all inconsistent states. The condition (LTS1) formalizes the backward propagation of inconsistencies, and (LTS2) captures the intuition that divergence should be viewed as catastrophic. A variant of the usual notion of weak ready simulation is recalled below, which is adopted to capture the refinement relation between processes in [3,4].

Definition 2.2 ([3])**.** Let $(P, Act_\tau, \rightarrow, F)$ be an LLTS. A relation $\mathcal{R} \subseteq P \times P$ is a stable ready simulation relation, if, for any $(p, q) \in \mathcal{R}$ and $a \in Act$, **(RS1)** both p and q are stable; **(RS2)** $p \notin F$ implies $q \notin F$; **(RS3)** $p \stackrel{a}{\Rightarrow}_F |p'$ implies $\exists q'.q \stackrel{a}{\Rightarrow}_F |q'$ and $(p', q') \in \mathcal{R}$; **(RS4)** $p \notin F$ implies $\mathcal{I}(p) = \mathcal{I}(q)$.

We say that p is stable ready simulated by q, in symbols $p \underset{\sim}{\sqsubset}_{RS} q$, if there exists a stable ready simulation relation $\mathcal{R}$ with $(p, q) \in \mathcal{R}$. Further, p is ready simulated by q, written $p \sqsubseteq_{RS} q$, if $\forall p'(p \stackrel{\epsilon}{\Rightarrow}_F |p'$ implies $\exists q'(q \stackrel{\epsilon}{\Rightarrow}_F |q'$ and $p' \underset{\sim}{\sqsubset}_{RS} q'))$. The kernels of $\underset{\sim}{\sqsubset}_{RS}$ and $\sqsubseteq_{RS}$ are denoted by $\approx_{RS}$ and $=_{RS}$ resp..

Next we fix some notations and terminologies related to CLL_R and recall some results obtained in [5]. Let V_{AR} be an infinite set of variables. Terms of CLL_R are given by the BNF grammar:

$$t ::= 0 \,|\perp|\, (\alpha.t) \mid (t \Box t) \mid (t \wedge t) \mid (t \vee t) \mid (t \parallel_A t) \mid X \mid \langle Z|E \rangle,$$

where $X \in V_{AR}$, $\alpha \in Act_\tau$, $A \subseteq Act$ and recursive specification $E = E(V)$ with $V \subseteq V_{AR}$ is a set of equations $\{Y = t : Y \in V\}$ and Z is a variable in V that

acts as the initial variable. We often denote $\langle X|\{X=t\}\rangle$ briefly by $\langle X|X=t\rangle$. In addition to standard operators in CCS and CSP, operators $\bot$, $\wedge$ and $\vee$ are introduced in CLL_R: $\bot$ represents an inconsistent process; $\vee$ and $\wedge$ are used to describe logical combinations of processes.

For any term $\langle Z|E\rangle$ with $E=E(V)$, each variable in V is bound with scope E. This induces the notion of free occurrence of variable, bound (and free) variables and α-equivalence as usual. The set of all processes (i.e., closed terms) is denoted by $T(\Sigma_{\mathrm{CLL}_R})$. We use p, q, r to represent processes. Throughout this note, we assume that recursive variables are distinct from each other and no recursive variable has free occurrence; moreover we don't distinguish between α-equivalent terms and use $\equiv$ for both syntactical identical and α-equivalence. For any t, the term $t\{\langle X|E\rangle/X : X \in V\}$ is denoted briefly by $\langle t|E\rangle$. A context $C_{\widetilde{X}}$ is a term whose free variables are in n-tuple distinct variables $\widetilde{X}=(X_1,...,X_n)$ with $n \geq 0$. Given $\widetilde{p}=(p_1,\dots,p_n)$, the term $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ is obtained from $C_{\widetilde{X}}$ by replacing X_i by p_i for each $i \leq n$ simultaneously.

Given a term t, a variable X is strongly (or weakly) guarded in t if each occurrence of X is within some subexpression $a.t_1$ ($\tau.t_1$ or $t_1 \vee t_2$ resp.). As usual, we assume that all recursive specifications (say $E(V)$) considered in the sequel are guarded (that is, for each $X \in V$ and $Z = t_Z \in E(V)$, each occurrence of X is within some subexpression $a.t_1$ or $\tau.t_1$ or $t_1 \vee t_2$).

SOS rules of CLL_R are divided into two parts: operational rules and predicate rules. Here we only list these rules in Table 1. For motivation behind these rules, we refer the reader to [5].

Operational rules

$$Ra_1\,\frac{-}{\alpha.x_1\stackrel{\alpha}{\rightarrow}x_1}\qquad Ra_2\,\frac{x_1\stackrel{a}{\rightarrow}y_1,\,x_2\stackrel{\tau}{\not\rightarrow}}{x_1\Box x_2\stackrel{a}{\rightarrow}y_1}\qquad Ra_3\,\frac{x_1\stackrel{\tau}{\not\rightarrow},\,x_2\stackrel{a}{\rightarrow}y_2}{x_1\Box x_2\stackrel{a}{\rightarrow}y_2}\qquad Ra_4\,\frac{x_1\stackrel{\tau}{\rightarrow}y_1}{x_1\Box x_2\stackrel{\tau}{\rightarrow}y_1\Box x_2}$$

$$Ra_5\,\frac{x_2\stackrel{\tau}{\rightarrow}y_2}{x_1\Box x_2\stackrel{\tau}{\rightarrow}x_1\Box y_2}\qquad Ra_6\,\frac{x_1\stackrel{a}{\rightarrow}y_1,\,x_2\stackrel{a}{\rightarrow}y_2}{x_1\wedge x_2\stackrel{a}{\rightarrow}y_1\wedge y_2}\qquad Ra_7\,\frac{x_1\stackrel{\tau}{\rightarrow}y_1}{x_1\wedge x_2\stackrel{\tau}{\rightarrow}y_1\wedge x_2}\qquad Ra_8\,\frac{x_2\stackrel{\tau}{\rightarrow}y_2}{x_1\wedge x_2\stackrel{\tau}{\rightarrow}x_1\wedge y_2}$$

$$Ra_9\,\frac{-}{x_1\vee x_2\stackrel{\tau}{\rightarrow}x_1}\qquad Ra_{10}\,\frac{-}{x_1\vee x_2\stackrel{\tau}{\rightarrow}x_2}\qquad Ra_{11}\,\frac{x_1\stackrel{\tau}{\rightarrow}y_1}{x_1\|_A x_2\stackrel{\tau}{\rightarrow}y_1\|_A x_2}\qquad Ra_{12}\,\frac{x_2\stackrel{\tau}{\rightarrow}y_2}{x_1\|_A x_2\stackrel{\tau}{\rightarrow}x_1\|_A y_2}$$

$$Ra_{13}\,\frac{x_1\stackrel{a}{\rightarrow}y_1,\,x_2\stackrel{\tau}{\not\rightarrow}}{x_1\|_A x_2\stackrel{a}{\rightarrow}y_1\|_A x_2}(a\notin A)\qquad Ra_{14}\,\frac{x_1\stackrel{\tau}{\not\rightarrow},\,x_2\stackrel{a}{\rightarrow}y_2}{x_1\|_A x_2\stackrel{a}{\rightarrow}x_1\|_A y_2}(a\notin A)$$

$$Ra_{15}\,\frac{x_1\stackrel{a}{\rightarrow}y_1,\,x_2\stackrel{a}{\rightarrow}y_2}{x_1\|_A x_2\stackrel{a}{\rightarrow}y_1\|_A y_2}(a\in A)\qquad Ra_{16}\,\frac{\langle t_X|E\rangle\stackrel{\alpha}{\rightarrow}y}{\langle X|E\rangle\stackrel{\alpha}{\rightarrow}y}(X=t_X\in E)$$

Predicative rules

$$Rp_1\,\frac{-}{\bot F}\qquad Rp_2\,\frac{x_1F}{\alpha.x_1F}\qquad Rp_3\,\frac{x_1F,\,x_2F}{x_1\vee x_2F}\qquad Rp_4\,\frac{x_1F}{x_1\Box x_2F}\qquad Rp_5\,\frac{x_2F}{x_1\Box x_2F}$$

$$Rp_6\,\frac{x_1F}{x_1\|_A x_2F}\qquad Rp_7\,\frac{x_2F}{x_1\|_A x_2F}\qquad Rp_8\,\frac{x_1F}{x_1\wedge x_2F}\qquad Rp_9\,\frac{x_2F}{x_1\wedge x_2F}$$

$$Rp_{10}\,\frac{x_1\stackrel{a}{\rightarrow}y_1,\,x_2\stackrel{a}{\not\rightarrow},\,x_1\wedge x_2\stackrel{\tau}{\not\rightarrow}}{x_1\wedge x_2F}\qquad Rp_{11}\,\frac{x_1\stackrel{a}{\not\rightarrow},\,x_2\stackrel{a}{\rightarrow}y_2,\,x_1\wedge x_2\stackrel{\tau}{\not\rightarrow}}{x_1\wedge x_2F}\qquad Rp_{12}\,\frac{x_1\wedge x_2\stackrel{\alpha}{\rightarrow}z,\,\{yF:x_1\wedge x_2\stackrel{\alpha}{\rightarrow}y\}}{x_1\wedge x_2F}$$

$$Rp_{13}\,\frac{\{yF:x_1\wedge x_2\stackrel{\epsilon}{\Rightarrow}|y\}}{x_1\wedge x_2F}\qquad Rp_{14}\,\frac{\langle t_X|E\rangle F}{\langle X|E\rangle F}(X=t_X\in E)\qquad Rp_{15}\,\frac{\{yF:\langle X|E\rangle\stackrel{\epsilon}{\Rightarrow}|y\}}{\langle X|E\rangle F}$$

Table 1: SOS rules of CLL_R

The calculus CLL_R has the unique stable transition model (denoted by M_{CLL_R}), which exactly consists of all positive literals of the form $t\stackrel{\alpha}{\rightarrow}t'$ or tF that are provable in $Strip(\mathrm{CLL}_R, M_{\mathrm{CLL}_R})$ [5]. Here $Strip(\mathrm{CLL}_R, M_{\mathrm{CLL}_R})$ is the

stripped version [1] of CLL_R w.r.t M_{CLL_R}. Each rule in $Strip(\mathrm{CLL}_R, M_{\mathrm{CLL}_R})$ is of the form $\frac{pprem(r)}{conc(r)}$ for some ground instance r of rules in Table 1 such that $M_{\mathrm{CLL}_R} \models nprem(r)$, where $nprem(r)$ (or, $pprem(r)$) is the set of negative (positive resp.) premises of r, $conc(r)$ is the conclusion of r and $M_{\mathrm{CLL}_R} \models nprem(r)$ means that for each $t \stackrel{\alpha}{\not\rightarrow} \in nprem(r)$, $t \stackrel{\alpha}{\rightarrow} s \notin M_{\mathrm{CLL}_R}$ for any s. The notion of proof tree in $Strip(\mathrm{CLL}_R, M_{\mathrm{CLL}_R})$ is defined as usual [1]. Notice that all proof trees are well-founded, and such fact will play central role in demonstrating the consistency of processes. Based on M_{CLL_R}, we can get the LTS $(T(\Sigma_{\mathrm{CLL}_R}), Act_\tau, \rightarrow_{\mathrm{CLL}_R}, F_{\mathrm{CLL}_R})$ ($LTS(\mathrm{CLL}_R)$ for short) in the standard way (e.g., [1]). For simplicity, we always omit the subscripts in $\stackrel{\alpha}{\rightarrow}_{\mathrm{CLL}_R}$ and F_{CLL_R}. We end this section by recalling some fundamental properties of $LTS(\mathrm{CLL}_R)$, which are asserted by Theorems 4.1 and 6.1 and Lemma 4.2 in [5].

Theorem 2.3. *(1) $LTS(CLL_R)$ is a τ-pure LLTS. (2) If $p \in F$ and $\tau \in \mathcal{I}(p)$ then $\forall q(p \stackrel{\tau}{\rightarrow} q$ implies $q \in F)$, and hence $p \stackrel{\epsilon}{\Rightarrow} |q$ and $q \notin F$ implies $p \stackrel{\epsilon}{\Rightarrow}_F |q$. (3) If $p \sqsubseteq_{RS} q$ then $C_X\{p/X\} \sqsubseteq_{RS} C_X\{q/X\}$ for any C_X, and hence, if $p \sqsubseteq_{RS} q$ and $C_X\{p/X\} \notin F$ then $C_X\{q/X\} \notin F$.*

3. Main results

In [5], the following theorem has been obtained.

Theorem 3.1 (Unique solution)**.** *For any $p, q \notin F$ and t_X where X is strongly guarded and does not occur in the scope of any conjunction, if $p =_{RS} t_X\{p/X\}$ and $q =_{RS} t_X\{q/X\}$ then $p =_{RS} q$. Moreover $\langle X|X = t_X\rangle$ is the unique consistent solution (modulo $=_{RS}$) of the equation $X =_{RS} t_X$ whenever consistent solutions exist.*

The next example demonstrates that this theorem no longer holds if we drop the assumption that X does not occur in the scope of any conjunction.

Example 3.2. Consider the equation $X =_{RS} t_X$ where $t_X \equiv (\langle Y|Y = a.Y\rangle \wedge a.X) \vee (\langle Z|Z = b.Z\rangle \wedge b.X)$. Clearly, X is strongly guarded in t_X. We shall show that both $\langle X|X = a.X\rangle$ and $\langle X|X = b.X\rangle$ are consistent solutions.

Let us first prove that $\langle X|X = a.X\rangle \notin F$. On the contrary, suppose that $\langle X|X = a.X\rangle \in F$. Then the last rule applied in the proof tree of $\langle X|X = a.X\rangle F$ is either $\frac{a.\langle X|X=a.X\rangle F}{\langle X|X=a.X\rangle F}$ or $\frac{\{rF:\langle X|X=a.X\rangle \stackrel{\epsilon}{\Rightarrow}|r\}}{\langle X|X=a.X\rangle F}$. Then it is easy to see that every proof tree of $\langle X|X = a.X\rangle F$ has a proper subtree with root $\langle X|X = a.X\rangle F$, this contradicts the well-foundedness of proof tree, as desired.

Secondly we show that $\langle X|X = a.X\rangle$ is a solution. Analysis similar to that above shows that $\langle Y|Y = a.Y\rangle \wedge a.\langle X|X = a.X\rangle \notin F$ and $\langle Y|Y = a.Y\rangle \wedge \langle X|X = a.X\rangle \notin F$. Then it is easy to check that the binary relation $\mathcal{R}$ given below is a stable ready simulation relation, where $P_v \triangleq \langle v|v = a.v\rangle$ with $v \in \{X, Y\}$.

$$\mathcal{R} \triangleq \{(P_X, P_Y \wedge a.P_X), (P_X, P_Y \wedge P_X), (P_Y \wedge a.P_X, P_X), (P_Y \wedge P_X, P_X)\}.$$

Hence

$$\langle X|X = a.X\rangle =_{RS} \langle Y|Y = a.Y\rangle \wedge a.\langle X|X = a.X\rangle. \tag{3.2.1}$$

Moreover, $\langle Z|Z = b.Z\rangle \wedge b.\langle X|X = a.X\rangle \in F$ by Rules Rp_{10}, Rp_{11} and Rp_{12}, which, together with (3.2.1), implies $\langle X|X = a.X\rangle =_{RS} t_X\{\langle X|X = a.X\rangle/X\}$.

Summarily, $\langle X|X = a.X\rangle$ is a consistent solution. Similarly, so is $\langle X|X = b.X\rangle$. However, $\langle X|X = a.X\rangle \neq_{RS} \langle X|X = b.X\rangle$. □

For any equation $X =_{RS} t_X$, it is obvious that $\langle X|X = t_X\rangle$ is a solution of this equation. Moreover, the preceding example reveals that there may be more than one (consistent) solution. Then it is natural to try to relate $\langle X|X = t_X\rangle$ to other solutions. As the main result of this note, we intend to show that, if X is strongly guarded in t_X then $\langle X|X = t_X\rangle$ is the greatest solution of the equation $X =_{RS} t_X$. In other words, $\langle X|X = t_X\rangle$ captures the loosest solution whenever X is strongly guarded in t_X. To this end, a few of results in [5] are recalled below. The following facts are confirmed by Lemmas 5.6-5.8 in [5].

Lemma 3.3. *If $C_X\{p/X\} \xrightarrow{\alpha} r$ then*

(1) if $\alpha = \tau$ then either (1.1) there exists C'_X such that $r \equiv C'_X\{p/X\}$ and $C_X\{q/X\} \xrightarrow{\tau} C'_X\{q/X\}$ for any q, or (1.2) there exist $C'_{X,Z}$ and p' such that $p \xrightarrow{\tau} p'$, $r \equiv C'_{X,Z}\{p/X, p'/Z\}$ and $C_X\{q/X\} \xrightarrow{\tau} C'_{X,Z}\{q/X, q'/Z\}$ for any $q \xrightarrow{\tau} q'$;

(2) if $\alpha \in Act$ then there exits $C'_{X,\widetilde{Y}}$ such that (2.1) $r \equiv C'_{X,\widetilde{Y}}\{p/X, \widetilde{p'_Y}/\widetilde{Y}\}$ for some $\widetilde{p'_Y}$ with $p \xrightarrow{\alpha} p'_Y$ for each $Y \in \widetilde{Y}$; (2.2) if $C_X\{q/X\}$ is stable and $q \xrightarrow{\alpha} q'_Y$ for each $Y \in \widetilde{Y}$, then $C_X\{q/X\} \xrightarrow{\alpha} C'_{X,\widetilde{Y}}\{q/X, \widetilde{q'_Y}/\widetilde{Y}\}$;

(3) in particular, if X is guarded in C_X then there exists B_X such that $r \equiv B_X\{p/X\}$ and for any q, $C_X\{q/X\} \xrightarrow{\alpha} B_X\{q/X\}$. □

The next property is asserted by Lemmas 5.6, 5.8 and 5.14 in [5].

Lemma 3.4. *If $C_X\{p/X\} \overset{\epsilon}{\Rightarrow} |r$ then there exist $C'_{X,\widetilde{Y}}$ and p'_Y for $Y \in \widetilde{Y}$ such that (1) $p \overset{\tau}{\Rightarrow} |p'_Y$ for each $Y \in \widetilde{Y}$ and $r \equiv C'_{X,\widetilde{Y}}\{p/X, \widetilde{p'_Y}/\widetilde{Y}\}$; (2) for any q such that $q \xrightarrow{\tau}$ iff $p \xrightarrow{\tau}$, if $q \overset{\tau}{\Rightarrow} |q'_Y$ for each $Y \in \widetilde{Y}$ then $C_X\{q/X\} \overset{\epsilon}{\Rightarrow} |C'_{X,\widetilde{Y}}\{q/X, \widetilde{q'_Y}/\widetilde{Y}\}$; (3) in particular, if X is strongly guarded in C_X then so it is in $C'_{X,\widetilde{Y}}$, $\widetilde{Y} = \emptyset$ and $C_X\{q/X\} \overset{\epsilon}{\Rightarrow} |C'_{X,\widetilde{Y}}\{q/X\}$ for any q.* □

Lemma 3.5. *If X is strongly guarded in t_X and $p \sqsubseteq_{RS} t_X\{p/X\}$ then for any context C_Y, $C_Y\{t_X\{p/X\}/Y\} \notin F$ implies $C_Y\{\langle X|X = t_X\rangle/Y\} \notin F$.*

Proof. By Lemma 3.3(3) and Ra_{16}, we have $\mathcal{I}(\langle X|X = t_X\rangle) = \mathcal{I}(t_X\{p/X\})$. Then, by Lemma 3.3(1)(2), for any context D^*_Y, we get

$$\mathcal{I}(D^*_Y\{t_X\{p/X\}/Y\}) = \mathcal{I}(D^*_Y\{\langle X|X = t_X\rangle/Y\}). \tag{3.5.1}$$

Set $\Omega \triangleq \{B_Y\{\langle X|X = t_X\rangle/Y\} : B_Y$ is a context and $B_Y\{t_X\{p/X\}/Y\} \notin F\}$. To complete the proof, it suffices to prove that $F \cap \Omega = \emptyset$. We intend to show

that, for each $t \in \Omega$, any proof tree of tF has a proper subtree with root sF for some $s \in \Omega$. Such statement implies $F \cap \Omega = \emptyset$. Otherwise, a contradiction arises due to the fact that proof trees are well-founded. Let $\mathcal{T}$ be any proof tree of $C_Y\{\langle X|X = t_X\rangle/Y\}F$ with $C_Y\{\langle X|X = t_X\rangle/Y\} \in \Omega$. Then

$$C_Y\{t_X\{p/X\}/Y\} \notin F. \tag{3.5.2}$$

The rest of the proof runs by distinguishing cases based on C_Y. Here we handle only three non-trivial cases; the others are left to the reader.

Case 1. $C_Y \equiv Y$. Clearly, the last rule applied in $\mathcal{T}$ is either $\frac{\langle t_X|X=t_X\rangle F}{\langle X|X=t_X\rangle F}$ or $\frac{\{rF:\langle X|X=t_X\rangle \overset{\epsilon}{\Rightarrow}|r\}}{\langle X|X=t_X\rangle F}$. For the former, since $p \sqsubseteq_{RS} t_X\{p/X\}$, by (3.5.2) and Theorem 2.3(3), $t_X\{t_X\{p/X\}/X\} \notin F$. Hence $\mathcal{T}$ has a proper subtree with root $\langle t_X|X = t_X\rangle F$ and $\langle t_X|X = t_X\rangle \equiv t_X\{\langle X|X = t_X\rangle/X\} \in \Omega$, as desired.

For the latter, we treat the non-trivial case where $\langle X|X = t_X\rangle \overset{\tau}{\rightarrow}$. Since $t_X\{p/X\} \notin F$, by Theorem 2.3(1), $t_X\{p/X\} \overset{\epsilon}{\Rightarrow}_F |s$ for some s. For this transition, by Lemma 3.4(3), there exists t'_X such that $t_X\{\langle X|X = t_X\rangle/X\} \overset{\epsilon}{\Rightarrow} |t'_X\{\langle X|X = t_X\rangle/X\}$ and $s \equiv t'_X\{p/X\}$. Then, by Ra_{16} and $\langle X|X = t_X\rangle \overset{\tau}{\rightarrow}$, we get $\langle X|X = t_X\rangle \overset{\tau}{\Rightarrow} |t'_X\{\langle X|X = t_X\rangle/X\}$. So $\mathcal{T}$ has a proper subtree with root $t'_X\{\langle X|X = t_X\rangle/X\}F$. Moreover, by Theorem 2.3(3), we have $t'_X\{t_X\{p/X\}/X\} \notin F$ because of $s \equiv t'_X\{p/X\} \notin F$ and $p \sqsubseteq_{RS} t_X\{p/X\}$. Hence $t'_X\{\langle X|X = t_X\rangle/X\} \in \Omega$, as desired.

Case 2. $C_Y \equiv \langle Z|E\rangle$. Then the last rule applied in $\mathcal{T}$ is $\frac{\langle t_Z|E\rangle\{\langle X|X=t_X\rangle/Y\}F}{\langle Z|E\rangle\{\langle X|X=t_X\rangle/Y\}F}$ or $\frac{\{rF:\langle Z|E\rangle\{\langle X|X=t_X\rangle/Y\}\overset{\epsilon}{\Rightarrow}|r\}}{\langle Z|E\rangle\{\langle X|X=t_X\rangle/Y\}F}$. For the former, we get $\langle t_Z|E\rangle\{t_X\{p/X\}/Y\} \notin F$ due to Rp_{14} and (3.5.2). So $\langle t_Z|E\rangle\{\langle X|X = t_X\rangle/Y\} \in \Omega$, as desired.

For the latter, by (3.5.2) and Theorem 2.3(1), $C_Y\{t_X\{p/X\}/Y\} \overset{\epsilon}{\Rightarrow}_F |s$ for some s. For this transition, there exist $C'_{Y,\widetilde{W}}$ and $\widetilde{s'_W}$ that satisfy clauses (1,2,3) in Lemma 3.4. Hence $s \equiv C'_{Y,\widetilde{W}}\{t_X\{p/X\}/Y, \widetilde{s'_W}/\widetilde{W}\}$ and for each $W \in \widetilde{W}$, $t_X\{p/X\} \overset{\tau}{\Rightarrow} |s'_W$. For each such transition, say $t_X\{p/X\} \overset{\tau}{\Rightarrow} |s'_W$, by Lemma 3.4(3) and 3.3(3), $t_X\{\langle X|X = t_X\rangle/X\} \overset{\tau}{\Rightarrow} |t'^W_X\{\langle X|X = t_X\rangle/X\}$ and $s'_W \equiv t'^W_X\{p/X\}$ for some t'^W_X. So, $\langle X|X = t_X\rangle \overset{\tau}{\Rightarrow} |t'^W_X\{\langle X|X = t_X\rangle/X\}$ for each $W \in \widetilde{W}$. Further, since $C'_{Y,\widetilde{W}}$ satisfies clause (2) in Lemma 3.4, by (3.5.1) with $D^*_Y \equiv Y$, we get $C_Y\{\langle X|X = t_X\rangle/Y\} \overset{\epsilon}{\Rightarrow} |u$, where $u \overset{\triangle}{\equiv} C'_{Y,\widetilde{W}}\{\langle X|X = t_X\rangle/Y, \widetilde{t'^W_X}\{\langle X|X = t_X\rangle/X\}/\widetilde{W}\}$. Hence $\mathcal{T}$ has a proper subtree with root uF. Moreover, since $s \equiv C'_{Y,\widetilde{W}}\{t_X\{p/X\}/Y, \widetilde{t'^W_X\{p/X\}}/\widetilde{W}\} \notin F$ and $p \sqsubseteq_{RS} t_X\{p/X\}$, we obtain $C'_{Y,\widetilde{W}}\{t_X\{p/X\}/Y, \widetilde{t'^W_X}\{t_X\{p/X\}/X\}/\widetilde{W}\} \notin F$ due to Theorem 2.3(3). Then $u \in \Omega$, as desired.

Case 3. $C_Y \equiv B_Y \wedge D_Y$. We distinguish four cases based on the last rule applied in $\mathcal{T}$. Since rules for $\wedge$ are symmetric w.r.t its operands, we consider only one of two symmetric rules.

Case 3.1. $\frac{B_Y\{\langle X|X=t_X\rangle/Y\}F}{C_Y\{\langle X|X=t_X\rangle/Y\}F}$. By (3.5.2) and Rp_8, $B_Y\{t_X\{p/X\}/Y\} \notin F$ and

hence $B_Y\{\langle X|X=t_X\rangle/Y\} \in \Omega$.

Case 3.2. $\frac{B_Y\{\langle X|X=t_X\rangle/Y\}\xrightarrow{\alpha} r}{C_Y\{\langle X|X=t_X\rangle/Y\}F}$ with $D_Y\{\langle X|X=t_X\rangle/Y\} \overset{a}{\nrightarrow}$ and $C_Y\{\langle X|X=t_X\rangle/Y\} \overset{\tau}{\nrightarrow}$. Then, by (3.5.1), we get $B_Y\{t_X\{p/X\}/Y\} \xrightarrow{\alpha}$, $D_Y\{t_X\{p/X\}/Y\} \overset{a}{\nrightarrow}$ and $C_Y\{t_X\{p/X\}/Y\} \overset{\tau}{\nrightarrow}$. Thus $C_Y\{t_X\{p/X\}/Y\} \in F$ follows by Rp_{10} and Rp_{11}, which contradicts (3.5.2). Hence this case is impossible.

Case 3.3. $\frac{\{rF : C_Y\{\langle X|X=t_X\rangle/Y\}\overset{\epsilon}{\Rightarrow}|r\}}{C_Y\{\langle X|X=t_X\rangle/Y\}F}$. Similar to the second alternative in the proof of Case 2, omitted.

Case 3.4. $\frac{C_Y\{\langle X|X=t_X\rangle/Y\}\xrightarrow{\alpha} r', \{rF : C_Y\{\langle X|X=t_X\rangle/Y\}\xrightarrow{\alpha} r\}}{C_Y\{\langle X|X=t_X\rangle/Y\}F}$. Then, by (3.5.1) with $D^*_Y \equiv C_Y$, (3.5.2) and Theorem 2.3(1), there exists s such that

$$C_Y\{t_X\{p/X\}/Y\} \xrightarrow{\alpha}_F s. \tag{3.5.3}$$

In the following, we consider two cases based on α.

Case 3.4.1. $\alpha = \tau$. For the transition in (3.5.3), either (1.1) or (1.2) in Lemma 3.3 holds. For the former, there exists C'_Y such that $C_Y\{\langle X|X=t_X\rangle/Y\} \xrightarrow{\tau} C'_Y\{\langle X|X=t_X\rangle/Y\}$ and $s \equiv C'_Y\{t_X\{p/X\}/Y\}$. Then $\mathcal{T}$ has a proper subtree with root $C'_Y\{\langle X|X=t_X\rangle/Y\}F$ and $C'_Y\{\langle X|X=t_X\rangle/Y\} \in \Omega$ due to $s \equiv C'_Y\{t_X\{p/X\}/Y\} \notin F$.

Next we handle the latter where (1.2) in Lemma 3.3 holds. In such situation, $s \equiv C'_{Y,Z}\{t_X\{p/X\}/Y, s'/Z\}$ for some $s', C'_{Y,Z}$ such that $t_X\{p/X\} \xrightarrow{\tau} s'$ and

$$C_Y\{q/Y\} \xrightarrow{\tau} C'_{Y,Z}\{q/Y, q'/Z\} \text{ for any } q \xrightarrow{\tau} q'. \tag{3.5.4}$$

For $t_X\{p/X\} \xrightarrow{\tau} s'$, by Lemma 3.3(3), there exists t'_X such that $s' \equiv t'_X\{p/X\}$ and $t_X\{\langle X|X=t_X\rangle/X\} \xrightarrow{\tau} t'_X\{\langle X|X=t_X\rangle/X\}$. Then by Ra_{16}, $\langle X|X=t_X\rangle \xrightarrow{\tau} t'_X\{\langle X|X=t_X\rangle/X\}$. Further, it follows from (3.5.4) that $C_Y\{\langle X|X=t_X\rangle/Y\} \xrightarrow{\tau} u$, where $u \triangleq C'_{Y,Z}\{\langle X|X=t_X\rangle/Y, t'_X\{\langle X|X=t_X\rangle/X\}/Z\}$. Thus $\mathcal{T}$ has a proper subtree with root uF. Moreover, by Theorem 2.3(3) and $s \notin F$, we get $C'_{Y,Z}\{t_X\{p/X\}/Y, t'_X\{t_X\{p/X\}/X\}/Z\} \notin F$, which implies $u \in \Omega$.

Case 3.4.2. $\alpha \in Act$. For the transition in (3.5.3), there exists $C'_{Y,\widetilde{Z}}$ that satisfies (2.1) and (2.2) in Lemma 3.3(2). Thus $s \equiv C'_{Y,\widetilde{Z}}\{t_X\{p/X\}/Y, \widetilde{s'_Z}/\widetilde{Z}\}$ for some $\widetilde{s'_Z}$ with $t_X\{p/X\} \xrightarrow{\alpha} s'_Z$ for any $Z \in \widetilde{Z}$. For each such transition, say $t_X\{p/X\} \xrightarrow{\alpha} s'_Z$, by Lemma 3.3(3), $s'_Z \equiv t'^Z_X\{p/X\}$ and $t_X\{\langle X|X=t_X\rangle/X\} \xrightarrow{\alpha} t'^Z_X\{\langle X|X=t_X\rangle/X\}$ for some t'^Z_X. Then, by Ra_{16}, $\langle X|X=t_X\rangle \xrightarrow{\alpha} t'^Z_X\{\langle X|X=t_X\rangle/X\}$ for $Z \in \widetilde{Z}$. Further, since $C'_{Y,\widetilde{Z}}$ satisfies (2.2) in Lemma 3.3, by (3.5.1) with $D^*_Y \equiv C_Y$, we get $C_Y\{\langle X|X=t_X\rangle/Y\} \xrightarrow{\alpha} u$, where $u \triangleq C'_{Y,\widetilde{Z}}\{\langle X|X=t_X\rangle/Y, \widetilde{t'^Z_X}\{\langle X|X=t_X\rangle/X\}/\widetilde{Z}\}$. Thus $\mathcal{T}$ has a proper subtree with root uF. Moreover, by $s \notin F$ and Theorem 2.3(3), we get $C'_{Y,\widetilde{Z}}\{t_X\{p/X\}/Y, \widetilde{t'^Z_X}\{t_X\{p/X\}/X\}/\widetilde{Z}\} \notin F$, and hence $u \in \Omega$. □

Having disposed of this preliminary step, we can now give a crucial result.

Let us first recall a notion of up-to $\underset{\sim RS}{\sqsubset}$, which depends on an equivalent formulation of $\sqsubseteq_{RS}$ provided by van Glabbeek [3].

Definition 3.6 ([5])**.** A relation $\mathcal{R} \subseteq T(\Sigma_{\mathrm{CLL}_R}) \times T(\Sigma_{\mathrm{CLL}_R})$ is a ready simulation relation up to $\underset{\sim RS}{\sqsubset}$ whenever, for any $(p,q) \in \mathcal{R}$ and $a \in Act$,

(Upto-1) $p \overset{\epsilon}{\Rightarrow}_F |p'$ implies $\exists q'.q \overset{\epsilon}{\Rightarrow}_F |q'$ and $p' \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset} q'$;

(Upto-2) $p \overset{a}{\Rightarrow}_F |p'$ and p, q stable implies $\exists q'.q \overset{a}{\Rightarrow}_F |q'$ and $p' \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset} q'$;

(Upto-3) $p \notin F$ and p, q stable implies $\mathcal{I}(p) = \mathcal{I}(q)$.

This notion provides a sound up-to technique, that is, if $\mathcal{R}$ is a ready simulation relation up to $\underset{\sim RS}{\sqsubset}$, then $\mathcal{R} \subseteq \sqsubseteq_{RS}$ [5]. The next lemma asserts that $\langle X|X = t_X\rangle$ is the largest solution of the inequation $X \sqsubseteq_{RS} t_X$.

Lemma 3.7. *If $p \sqsubseteq_{RS} t_X\{p/X\}$ then $p \sqsubseteq_{RS} \langle X|X = t_X\rangle$ whenever X is strongly guarded in t_X.*

Proof. By Lemma 3.3(3) and Ra_{16}, we get $\mathcal{I}(t_X\{p/X\}) = \mathcal{I}(\langle X|X = t_X\rangle)$. Then, by Lemma 3.3(1)(2), it follows that, for any D_Y,

$$\mathcal{I}(D_Y\{t_X\{p/X\}/Y\}) = \mathcal{I}(D_Y\{\langle X|X = t_X\rangle/Y\}). \tag{3.7.1}$$

To complete the proof, it suffices to show that $t_X\{p/X\} \sqsubseteq_{RS} \langle X|X = t_X\rangle$. Set $\mathcal{R} \triangleq \{(B_Y\{t_X\{p/X\}/Y\}, B_Y\{\langle X|X = t_X\rangle/Y\}) : B_Y \text{ is a context}\}$. We intend to prove that $\mathcal{R}$ is a ready simulation relation up to $\underset{\sim RS}{\sqsubset}$.

Let $(C_Y\{t_X\{p/X\}/Y\}, C_Y\{\langle X|X = t_X\rangle/Y\}) \in \mathcal{R}$. We shall check that such pair satisfies (Upto-1,2,3). For (Upto-3), it is obvious due to (3.7.1).

(Upto-1) Assume $C_Y\{t_X\{p/X\}/Y\} \overset{\epsilon}{\Rightarrow}_F |s$. So there exist $C'_{Y,\widetilde{Z}}$ and $\widetilde{s'_Z}$ that satisfy clauses (1)-(3) in Lemma 3.4. Then $t_X\{p/X\} \overset{\tau}{\Rightarrow} |s'_Z$ for $Z \in \widetilde{Z}$ and $s \equiv C'_{Y,\widetilde{Z}}\{t_X\{p/X\}/Y, \widetilde{s'_Z}/\widetilde{Z}\}$. For each such transition, say $t_X\{p/X\} \overset{\tau}{\Rightarrow} |s'_Z$, by Lemma 3.4(3) and 3.3(3), there exists t'^Z_X with strongly guarded X such that $s'_Z \equiv t'^Z_X\{p/X\}$ and $t_X\{\langle X|X = t_X\rangle/X\} \overset{\tau}{\Rightarrow} |t'^Z_X\{\langle X|X = t_X\rangle/X\}$. So, by Ra_{16}, $\langle X|X = t_X\rangle \overset{\tau}{\Rightarrow} |t'^Z_X\{\langle X|X = t_X\rangle/X\}$ for $Z \in \widetilde{Z}$. Since $C'_{Y,\widetilde{Z}}$ satisfies clause (2) in Lemma 3.4, by (3.7.1) with $D_Y \equiv Y$, we get $C_Y\{\langle X|X = t_X\rangle/Y\} \overset{\epsilon}{\Rightarrow} |u$, where $u \triangleq C'_{Y,\widetilde{Z}}\{\langle X|X = t_X\rangle/Y, \widetilde{t'^Z_X\{\langle X|X = t_X\rangle/X\}}/\widetilde{Z}\}$. Put $w \triangleq C'_{Y,\widetilde{Z}}\{t_X\{p/X\}/Y, \widetilde{t'^Z_X\{t_X\{p/X\}/X\}}/\widetilde{Z}\}$. Since $p \sqsubseteq_{RS} t_X\{p/X\}$, by Theorem 2.3(3) and $s \equiv C'_{Y,\widetilde{Z}}\{t_X\{p/X\}/Y, \widetilde{t'^Z_X\{p/X\}}/\widetilde{Z}\} \notin F$, we get $s \sqsubseteq_{RS} w$ and hence $w \notin F$. So $u \notin F$ by Lemma 3.5. Since $C_Y\{\langle X|X = t_X\rangle/Y\} \overset{\epsilon}{\Rightarrow} |u$, by Theorem 2.3(2), we get $C_Y\{\langle X|X = t_X\rangle/Y\} \overset{\epsilon}{\Rightarrow}_F |u$. Moreover, since X is strongly guarded in t'^Z_X for $Z \in \widetilde{Z}$, X is strongly guarded in $C'_{Y,\widetilde{Z}}\{t_X\{p/X\}/Y, \widetilde{t'^Z_X}/\widetilde{Z}\}$. So $w \equiv C'_{Y,\widetilde{Z}}\{t_X\{p/X\}/Y, \widetilde{t'^Z_X}/\widetilde{Z}\}\{t_X\{p/X\}/X\} \overset{\tau}{\not\to}$ due to Lemma 3.3(3) and $s \equiv C'_{Y,\widetilde{Z}}\{t_X\{p/X\}/Y, \widetilde{t'^Z_X}/\widetilde{Z}\}\{p/X\} \overset{\tau}{\not\to}$. Thus $s \underset{\sim RS}{\sqsubset} w \mathcal{R} u$ because of $s \sqsubseteq_{RS} w$.

(Upto-2) Let $C_Y\{t_X\{p/X\}/Y\}$ and $C_Y\{\langle X|X = t_X\rangle/Y\}$ be stable, and let $C_Y\{t_X\{p/X\}/Y\} \overset{a}{\Rightarrow}_F |s$. Then $C_Y\{t_X\{p/X\}/Y\} \overset{a}{\rightarrow}_F r \overset{\epsilon}{\Rightarrow}_F |s$ for some r. For the transition $C_Y\{t_X\{p/X\}/Y\} \overset{a}{\rightarrow} r$, there exists $C'_{Y,\widetilde{Z}}$ that satisfies clauses (2.1) and (2.2) in Lemma 3.3. Then $r \equiv C'_{Y,\widetilde{Z}}\{t_X\{p/X\}/Y, \widetilde{r'_Z}/\widetilde{Z}\}$ for some $\widetilde{r'_Z}$ such that $t_X\{p/X\} \overset{a}{\rightarrow} r'_Z$ for $Z \in \widetilde{Z}$. For each such transition, say $t_X\{p/X\} \overset{a}{\rightarrow} r'_Z$, by Lemma 3.3(3), $r'_Z \equiv t'^Z_X\{p/X\}$ and $t_X\{\langle X|X = t_X\rangle/X\} \overset{a}{\rightarrow} t'^Z_X\{\langle X|X = t_X\rangle/X\}$ for some t'^Z_X. Then, by Ra_{16}, $\langle X|X = t_X\rangle \overset{a}{\rightarrow} t'^Z_X\{\langle X|X = t_X\rangle/X\}$ for $Z \in \widetilde{Z}$. Further, since $C'_{Y,\widetilde{Z}}$ satisfies clause (2.2) in Lemma 3.3, we get $C_Y\{\langle X|X = t_X\rangle/Y\} \overset{a}{\rightarrow} v$, where $v \triangleq C'_{Y,\widetilde{Z}}\{\langle X|X = t_X\rangle/Y, \widetilde{t'^Z_X}\{\langle X|X = t_X\rangle/X\}/\widetilde{Z}\}$. Let $u \triangleq C'_{Y,\widetilde{Z}}\{t_X\{p/X\}/Y, \widetilde{t'^Z_X}\{t_X\{p/X\}/X\}/\widetilde{Z}\}$. By Theorem 2.3(3), we have $r \equiv C'_{Y,\widetilde{Z}}\{t_X\{p/X\}/Y, \widetilde{t'^Z_X\{p/X\}}/\widetilde{Z}\} \sqsubseteq_{RS} u$ because of $p \sqsubseteq_{RS} t_X\{p/X\}$. Further, it follows from $r \overset{\epsilon}{\Rightarrow}_F |s$ that $u \overset{\epsilon}{\Rightarrow}_F |t$ and $s \underset{\sim}{\sqsubset}_{RS} t$ for some t. Since $u\mathcal{R}v$, by (Upto-1), there exits t' such that $v \overset{\epsilon}{\Rightarrow}_F |t'$ and $t \underset{\sim}{\sqsubset}_{RS} \mathcal{R} \underset{\sim}{\sqsubset}_{RS} t'$. Moreover, by Lemma 3.5, $C_Y\{\langle X|X = t_X\rangle/Y\} \notin F$ due to $C_Y\{t_X\{p/X\}/Y\} \notin F$. Hence $C_Y\{\langle X|X = t_X\rangle/Y\} \overset{a}{\rightarrow}_F v \overset{\epsilon}{\Rightarrow}_F |t'$ and $s \underset{\sim}{\sqsubset}_{RS} t \underset{\sim}{\sqsubset}_{RS} \mathcal{R} \underset{\sim}{\sqsubset}_{RS} t'$, as desired. □

As a consequence of Lemma 3.7 and Theorem 2.3(3), our main result is arrived, which characterizes $\langle X|X = t_X\rangle$ as the greatest solution of $X =_{RS} t_X$.

Theorem 3.8. *For any t_X with strongly guarded X, $\langle X|X = t_X\rangle$ is the greatest solution (w.r.t $\sqsubseteq_{RS}$) of $X =_{RS} t_X$; moreover $\langle X|X = t_X\rangle$ is consistent iff consistent solutions exit.*

We give a brief discussion to conclude this note. For Theorem 3.8, the hypothesis that X is strongly guarded cannot be relaxed to that X is weakly guarded. For instance, consider the equation $X =_{RS} \tau.X$, since $p =_{RS} \tau.p$ always holds for any p, such equation has infinitely many consistent solutions. However, since $\langle X|X = \tau.X\rangle$ is inconsistent by Theorem 2.3(1) and (LTS2) in Definition 2.1, it is the least solution of the equation $X =_{RS} \tau.X$.

References

[1] R. Bol, J.F. Groote, The meaning of negative premises in transition system specifications, JACM 43 (1996) 863-914.

[2] G. Lüttgen, W. Vogler, Conjunction on processes: full-abstraction via ready-tree semantics, TCS 373 (1-2) (2007) 19-40.

[3] G. Lüttgen, W. Vogler, Ready simulation for concurrency: it's logical, Inform. & comp. 208 (2010) 845-867.

[4] G. Lüttgen, W. Vogler, Safe reasoning with Logic LTS, TCS 412 (2011) 3337-3357.

[5] Y. Zhang, Z.H. Zhu, J.J. Zhang, On recursive operations over logic LTS, MSCS, available on CJO 2014 doi:10.1017/S0960129514000073.